Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T-A)$$

Where A is the temperature of the surrounding environment, and -k is the constant of proportionality.

$$T = Ce^{-kt} + A$$

Solving First Order Linear Eqs.

- Given an FODE of the form
  $$a_1(x)y' + a_2(x)y = f(x)$$
  Divide both sides by $a_1(x)$ to get an equation of the standard form
  $$y' + P(x)y = q(x)$$
  Then multiply both sides by the integrating factor
  $$I(x) = e^{\int P(x)dx}$$
  To reverse product rule and then integrate both sides with respect to x.

Mixing Problems

$$\frac{dA}{dt} = \left(\frac{conentration}{in} * \frac{rate}{in}\right) - \left(\frac{concentration}{out} * \frac{rate}{out}\right)$$

Note: The net volume might be changing so make sure to reflect that in the concentration out field.

Second Order Linear Eqs.

- Given a homogenous SODE of the form
  $$a_2(x)y'' + a_1(x)y' + a_2(x)y = 0$$
  The principle of superposition states that if $y_1(x)$ and $y_2(x)$ are linearly independent solutions, then $y = c_1y_1(x) + c_2y_2(x)$ is also a solution.
- Given a homogenous SODE of Constant Coefficients
  $$ay'' + by' + cy = 0$$
  Its characteristic equation is given by
  $$ar^2 + br + c = 0$$
  And solving for *r* can help us determine the form of the solution.
  $$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
  - Case 1: $b^2 - 4ac > 0$ → roots are real and distinct
    - General Solution: $y = c_1e^{r_1x} + c_2e^{r_2x}$
  - Case 2: $b^2 - 4ac < 0$ → roots complex
    $$r = \alpha \pm \beta i$$
    - General Solution: $y = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$
  - Case 1: $b^2 - 4a => 0$ → roots are real and equal
    $$r = -\frac{b}{2a}$$
    - General Solution: $y = c_1e^{rx} + c_2xe^{rx}$

Springs

- Note: Displacements are positive in the downward direction and negative in the upward direction.
- The motion of a spring is modeled by the diff eq.
  $$mx'' + cx' + kx = f(t)$$
  Where *m* is the mass attached, c ≥ 0 is the damping constant, and k is the spring constant.
  - Case 1: Free Undamped Motion (*f(t)=0, c=0*)
    - $mx'' + kx = 0 \to x'' + \frac{k}{m}x = 0$
    - $w = \sqrt{\frac{k}{m}}$ (circular frequency)
    - $x(t) = c_1\cos(wt) + c_2\sin(wt)$
    - $A = \sqrt{c_1^2 + c_2^2}$
      - $\cos(\phi) = \frac{c_1}{A}$
      - $\sin(\phi) = \frac{c_2}{A}$
    - Phase angle ϕ:
      - 1st quadrant → $\phi = \arctan\left(\frac{c_1}{c_2}\right)$
      - 2nd quadrant → $\phi = \pi + \arctan\left(\frac{c_1}{c_2}\right)$
      - 3rd quadrant → $\phi = \pi + \arctan\left(\frac{c_1}{c_2}\right)$
      - 4nd quadrant → $\phi = 2\pi + \arctan\left(\frac{c_1}{c_2}\right)$
    - Amplitude-phase form:
      - $x(t) = Acos(wt - \phi)$
  - Case 2: Free Damped Motion (*f(t)=0, c>0*)
    - $mx'' + cx' + kx = 0$
      - Characteristic eq: $mr^2 + cr + k = 0$
      - $r = \frac{-c \pm \sqrt{c^2-4mk}}{2m}$
    - $c^2 - 4mk > 0$ → 2 real distinct roots
      - Motion is overdamped
    - $c^2 - 4mk = 0$ → 2 real equal roots
      - Motion is critically damped
    - $c^2 - 4mk < 0$ → 2 complex roots
      - Motion is underdamped
      - $r = p + \mu i$
      - $x(t) = e^{pt}(c_1\cos(\mu t) + c_2\sin(\mu t))$

      In amplitude-phase form
      - $x(t) = Ae^{pt}\cos(\mu t - \phi)$

        Where μ is the pseudo-frequency.

Nonhomogeneous Equations and Undetermined Coefficients

- The solution to the nonhomogeneous equation of the form
  $$ay'' + by' + cy = f(x)$$
  Is given by $y = y_c + y_p$, where y_c (called the complimentary solution) is the general solution to $ay'' + by' + cy = 0$, and y_p (called the particular solution is any function satisfying $ay'' + by' + cy = f(x)$.
- y_p can be found using the method of undetermined coefficients:

If $ay'' + by' + cy = f(x)$	Then the particular solution y_p is
1. *f(x)* = a polynomial in x	$y_p = x^k$(a general polynomial of the same degree) - where *k* is the number of times that 0 is a root of the characteristic equation
2. $f(x) = e^{ax}$(a polynomial in x)	$y_p = x^ke^{ax}$(a general polynomial of the same degree) - where *k* is the number of times that *a* is a root of the characteristic equation
3. $f(x) = e^{ax}\cos(bx)$ (a polynomial in x) Or 4. $f(x) = e^{ax}\cos(bx)$ (a polynomial in x)	$y_p = x^ke^{ax}$[(polynomial of same degree)cos(bx) + (another polynomial of same degree)sin(bx)] - where *k* is the number of times that $a \pm bi$ are the roots of the characteristic equation.

More Springs

- Case 3: Forced Damped motion (*f(t) ≠ 0, c > 0*)
  - Simply use method of undetermined coefficients to solve.
- In a solution to the diff. eq., the part that approaches 0 as *t* approaches ∞ is called the transient part of the solution, and the part that remains is called the steady-state part of the solution.
- Case 4: Forced Undamped Motion (Resonance, *f(t) ≠ 0, c = 0)*
  - Diff eq is of the form
    - $mx'' + kx = f(t) = \begin{cases} Fsin(w_1t) \\ Fcos(w_1t) \end{cases}$
    - Resonance occurs if the angle in the external force, w_1, is the same as the circular frequency, w, i.e. $w_1 = w = \sqrt{\frac{k}{m}}$
    - The graph is typically a wave with a constant frequency but increasing amplitude.

Laplace Transform

- The Laplace Transform is defined as
  $$L[f(t)] = \int_0^\infty e^{-st}f(t)dt$$
  And exists only provided that the improper integral converges.
- Linearity Property:
  - $L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$
- **Shifting Theorems**
  - If $L[f(t)] = F(s)$, then $L[e^{at}f(t)] = F(s-a)$
  - If $L[f(t)] = F(s)$, then
    - $L[u(t-a)f(t-a)] = e^{-as}F(s)$
    - $L^{-1}[e^{-as}F(s)] = u(t-a)f(t-a)$
  - $L[\delta(t-a)] = e^{-as}$
    - $\frac{d}{dt}u(t-a) = \delta(t-a)$

Inverses

- The $Adj(A) = C^T$, where *C* is the matrix of cofactors, c_{ij}, where $c_{ij} = (-1)^{i+j} \cdot \det(M_{ij})$
- $A^{-1} = \frac{1}{\det(A)}Adj(A)$

Eigenvectors

- For a complex eigenvalue and corresponding eigenvector, its conjugate is also an eigenvalue and all complex values in the eigenvector are replaced with their conjugate counterparts.

Eigenvalue Method for Linear Systems

- Given a homogenous linear system $X' = AX$, and A has eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots, v_n$, the general solution is given by
  $$x(t) = c_1e^{\lambda_1}v_1 + \cdots + c_ne^{\lambda_n}v_n$$
- Given a nonhomogeneous linear system $X' = AX + B$, solve for the general solution by finding X_c, the general solution to the homogeneous case, and then solving for X_p, which is the vector of undetermined coefficients, where an the entry in X_p corresponds to the same entry in B.
  - Once you set up X_p, take its derivative, X_p', and substitute it into the equation for X as
    $$X_p' = AX_p + B$$
    Then solve for the undetermined coefficients of X_p.
  - The solution then is $X = X_c + X_p$